Remarks on Product VMO

Michael T. Lacey[*]   Erin Terwilleger[†]
Georgia Institute of Technology   University of Connecticut

Brett D. Wick[‡]
Brown University

Abstract

Well known results related to the compactness of Hankel operators of one complex variable are extended to little Hankel operators of two complex variables. Critical to these considerations is the result of Ferguson and Lacey [5] characterizing the boundedness of the little Hankel operators in terms of the product BMO of S.-Y. Chang and R. Fefferman [2,3].

1   Introduction

We prove necessary and sufficient conditions for the compactness of little Hankel operators of two complex variables. In the one complex variable case, results of this type are sometimes referred to as Hartman's theorem. Central to this are the Hardy space and BMO of two complex variables. Formally, the easiest way to phrase these results is for the Hardy space $H^1(\mathbb{T} \otimes \mathbb{T})$ and its dual space BMO$(\mathbb{T} \otimes \mathbb{T})$. Definitions are postponed until the next section.

$L^2(\mathbb{T} \otimes \mathbb{T})$ is the direct sum of

$$L^2(\mathbb{T} \otimes \mathbb{T}) = \oplus_{\varepsilon \in \{\pm, \pm\}} H_\varepsilon^2(\mathbb{D} \otimes \mathbb{D})$$

in which $H_{\pm,\pm}^2(\mathbb{D} \otimes \mathbb{D})$ is the space of square integrable functions with (anti) holomorphic extensions in each variable separately. Let $\mathbb{P}_{\pm,\pm}$ be the corresponding projection of $L^2(\mathbb{T} \otimes \mathbb{T})$ onto $H_{\pm,\pm}^2(\mathbb{D} \otimes \mathbb{D})$.

The Hankel operators of interest to us are operators from $H_{+,+}^2(\mathbb{D} \otimes \mathbb{D})$ to $H_{-,-}^2(\mathbb{D} \otimes \mathbb{D})$ given by $h_\varphi := \mathbb{P}_{-,-} M_\varphi$ in which M_φ denotes the operator of pointwise multiplication by φ. The following theorem extends Nehari's Theorem [7] to two complex variables, and is essentially a restatement of the main result of S. Ferguson and the first author [5]. We indicate a modification of the classical proof, which relies in an essential way on the results of [5].

1.1 Theorem. *The Hankel operator h_φ is bounded iff there is a function $\psi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$ for which $\mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi$, and we have the equivalence*

(1.2)
$$\|h_\varphi\| \approx \inf\{\|\psi\|_\infty \,:\, \mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi\}$$

(1.3)
$$\approx \|\mathbb{P}_{-,-}\varphi\|_{\mathrm{BMO}(\mathbb{D} \otimes \mathbb{D})}.$$

The BMO space is the dual to real $H^1(\mathbb{D} \otimes \mathbb{D})$, as identified by S.-Y. Chang and R. Fefferman [2,3]. We have the following refinement of this theorem.

1.4 Corollary. *h_φ is compact iff $\mathbb{P}_{-,-}\varphi$ is in the closure of $C(\mathbb{T} \otimes \mathbb{T})$ with respect to the BMO topology.*

In view of the classical result of Sarason [9], we call this last space $\mathrm{VMO}(\mathbb{D} \otimes \mathbb{D})$. This space has an equivalent characterization in terms of Carleson measures. In the circumstance in which the symbol is assumed to be bounded, we can say a little more. Let $\mathcal{L}^p_{\pm,\pm}(\mathbb{T} \otimes \mathbb{T})$ be the space of functions $b \in L^p(\mathbb{T} \otimes \mathbb{T})$ such that $\mathbb{P}_{\pm,\pm}b = 0$.

1.5 Theorem. *Let $\varphi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$. Then the following are equivalent*

(i) h_φ is compact.

(ii) $\varphi \in \mathcal{L}^\infty_{-,-}(\mathbb{T} \otimes \mathbb{T}) + C(\mathbb{T} \otimes \mathbb{T})$.

(iii) there exists a $g \in C(\mathbb{T} \otimes \mathbb{T})$ such that $h_\varphi = h_g$.

This theorem is a consequence of a finer fact about the essential norm of a little Hankel operator. Take the essential norm to be

$$\|h_\varphi\|_e := \inf\{\|h_\varphi - K\| \,:\, K \,:\, H_{+,+}(\mathbb{T} \otimes \mathbb{T}) \to H_{-,-}(\mathbb{T} \otimes \mathbb{T}) \ \text{is compact}\}.$$

Observe that $\|h_\varphi\|_e = 0$ iff h_φ is compact.

1.6 Theorem. *Let $\varphi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$. Then*

$$\|h_\varphi\|_e \approx \mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-} + C).$$

These results have different, equivalent formulations in terms of Hankel matrices, or Hankel operators on $H^2(\mathbb{C}_+ \otimes \mathbb{C}_+)$. In addition, it is of interest to state a result in the equivalent language of commutators. Namely for a function $\varphi \in \mathrm{BMO}(\mathbb{R} \otimes \mathbb{R})$ define

(1.7)
$$C_\varphi := [[M_\varphi, H_1], H_2]$$

in which H_j denotes the Hilbert transform computed in the coordinate j. The Hilbert transform can be taken on the circle or the real line. At this point, we take it to be defined on the real line. Let us define

$$\mathrm{VMO}(\mathbb{R} \otimes \mathbb{R}) := \mathrm{clos}_{\mathrm{BMO}} C_0^\infty(\mathbb{R} \otimes \mathbb{R})$$

where C_0^∞ denotes the space of smooth compactly supported functions. We will return to the Carleson measure characterization of membership in VMO later.

1.8 Theorem. *We have* $\mathrm{VMO}(\mathbb{R} \otimes \mathbb{R})^* = H^1(\mathbb{R} \otimes \mathbb{R})$. *In addition,* C_φ *is compact iff* $\varphi \in \mathrm{VMO}(\mathbb{R} \otimes \mathbb{R})$.

The next section contains background material for this paper. Following that, the corollaries and theorems related to compact operators are given in sections three and four. The final section discusses the Carleson measure characterization of VMO, and the duality statement $\mathrm{VMO}^* = H^1$.

2 The Hardy Spaces of Two Complex Variables

In speaking of Hardy spaces, one should take care to specify whether the functions are analytic, or not. The analytic Hardy spaces $H^p(\mathbb{D} \otimes \mathbb{D})$ consists of functions $F : \mathbb{D} \otimes \mathbb{D} \to \mathbb{C}$ such that F is holomorphic in each variable seperately, and

$$\|F\|_p^p = \sup_{0 < r_j < 1} \int_{\mathbb{T} \otimes \mathbb{T}} |F(r_1 e^{2\pi i \theta_1}, r_2 e^{2\pi i \theta_2})|^p \, d\theta_1 \, d\theta_2.$$

In the case that $1 \leq p \leq \infty$, the boundry values of F exist almost everywhere. And the Fourier transform of that function is supported on the postive orthant of $\mathbb{Z} \otimes \mathbb{Z}$. In speaking of the analytic Hardy spaces, and their duals, we will use the notation $H^p(\mathbb{D} \otimes \mathbb{D})$, $H^p(\mathbb{C}_+ \otimes \mathbb{C}_+)$.

The (real) Hardy space $H^1(\mathbb{R} \otimes \mathbb{R})$ consists of real valued functions f on $\mathbb{R}^2$ for which

$$\|f\|_{H^1(\mathbb{R} \otimes \mathbb{R})} := \sum_{A_1, A_2 \in \{I, H_1, H_2\}} \|A_1 A_2 f\|_1 < \infty.$$

Here I is the identity operator and H_j is the Hilbert transform computed in the jth coordinate. For $f \in H^1(\mathbb{R} \otimes \mathbb{R})$, there is a biholomorphic extension $F(z_1, z_2)$ to $\mathbb{C}_+ \otimes \mathbb{C}_+$ such that

$$\lim_{y_1, y_2 \downarrow 0} \mathrm{Re}\, F(x_1 + iy_1, x_2 + iy_2) = f(x_1, x_2) \qquad \text{a.e.}$$

There are several equivalent definitions of this Hardy space in terms of maximal, square, and area functions, all formulated in terms of a product setting. In speaking of the real Hardy spaces, we will use the notation $H^1(\mathbb{R} \otimes \mathbb{R})$ or $H^1(\mathbb{T} \otimes \mathbb{T})$.

The dual space $\mathrm{BMO}(\mathbb{R} \otimes \mathbb{R})$ was identified by S.-Y. Chang and R. Fefferman. Their characterization is notable, as the structure of the allied Carleson measures is far more complicated than in a one parameter setting. This space has two known intrinsic characterizations. One is that $\mathrm{BMO}(\mathbb{R} \otimes \mathbb{R})$ is the dual to $H^1(\mathbb{R} \otimes \mathbb{R})$, and the second is that the BMO norm is comparable to the L^2 norm of the commutator C_b, which is one formulation of the main result of Ferguson and Lacey [5].

There is another definition in terms of wavelets and Carleson measures which, though no longer intrinsic in nature, is very useful. We let $\mathcal{D}$ denote the set of dyadic intervals on $\mathbb{R}$. Given a rectangle $R = R_1 \times R_2 \in \mathcal{D} \times \mathcal{D}$ define translation and dilation invariant operators by

$$T_y f(y) := f(x - y), \qquad y \in \mathbb{R}^2,$$

3

$$D^p_{R_1 \times R_2} f(x_1, x_2) := \frac{1}{(|R_1||R_2|)^{1/p}} f\left(\frac{x_1}{|R_1|}, \frac{x_2}{|R_2|}\right), \qquad 0 < p < \infty.$$

Note that the second condition preserves L^p norm and depends upon the scale but not location of the rectangle $R_1 \times R_2$.

Given a function $w(x_1, x_2) = \prod_1^2 v(x_j)$, we set

$$w_R = T_{c(R)} D^2_R w, \qquad c(R) = \text{ the center of } R.$$

Our assumptions are that v is a bounded, piecewise continuous, rapidly decreasing, mean zero function, and that $\{w_R : R \in \mathcal{D} \times \mathcal{D}\}$ is an L^2 normalized orthogonal basis for $L^2(\mathbb{R}^2)$.

Then, it is a theorem of Chang and Fefferman [2,3] that we have

(2.1)
$$\|f\|_{\text{BMO}(\mathbb{R} \otimes \mathbb{R})} \approx \sup_U \left[|U|^{-1} \sum_{R \subset U} |\langle f, w_R \rangle|^2 \right]^{1/2}.$$

What is essential in this definition is that the supremum be formed over all subsets U of the plane with finite measure.

To define analytic $\text{BMO}(\mathbb{C}_+ \otimes \mathbb{C}_+)$, one can use the same definition, provided one restricts attention to the jointly analytic projections of the wavelets. That is, the functions w_R are replaced by $v_R := \mathbb{P}_{+,+} w_R$, and then a definition of analytic BMO is just (2.1) with the w_R replaced by v_R.

By $A \lesssim B$ we mean that there is an absolute constant K so that $A \lesssim KB$. By $A \approx B$ we mean $A \lesssim B$ and $B \lesssim A$.

3 The Hankel Operators on $H^2(\mathbb{D} \otimes \mathbb{D})$

Proof of Theorem 1.1. If it is the case that $\psi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$ exists with $\mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi$, then clearly we can estimate

$$\|h_\varphi\| \leq \|\mathbb{P}_{-,-}\psi\|_2 \leq \|\psi\|_\infty.$$

It is also then the case that $\|\mathbb{P}_{-,-}\psi\|_{BMO(\mathbb{D} \otimes \mathbb{D})} \lesssim \|\psi\|_\infty$.

In the converse direction, we adopt a classical method of proof but use in an essential way the results of [5]. We show that there is a $\psi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$ with $\mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi$, and $\|\psi\|_\infty \lesssim \|h_\varphi\|$. We do so by defining a linear functional on $H^1(\mathbb{D} \otimes \mathbb{D})$ with norm less than or equal to a constant times $\|h_\varphi\|$. For a pair of functions $f, g \in H^2(\mathbb{D} \otimes \mathbb{D})$, set

$$L(fg) = \int (h_\varphi f) g \, dx = \int (P_{-,-}\varphi) fg \, dx.$$

It is important to observe that this definition does not depend upon the order in which f and g are given to us. And in addition, $|L(fg)| \leq \|h_\varphi\| \|f\|_{H^2} \|g\|_{H^2}$. Therefore, this definition of

L extends to the injective tensor product $H^2(\mathbb{D} \otimes \mathbb{D})\widehat{\otimes}H^2(\mathbb{D} \otimes \mathbb{D})$ which has the norm

$$\|h\|_{H^2\widehat{\otimes}H^2} := \inf\Big\{\sum_j \|f_j\|_{H^2}\|g_j\|_{H^2} \; : \; h = \sum_j f_j g_j\Big\}.$$

One way to phrase the main result of Ferguson and Lacey [5] is that we have the equality

$$H^2(\mathbb{D} \otimes \mathbb{D})\widehat{\otimes}H^2(\mathbb{D} \otimes \mathbb{D}) = H^1(\mathbb{D} \otimes \mathbb{D}).$$

Hence, the linear functional L extends to a bounded linear functional on $H^1(\mathbb{D} \otimes \mathbb{D})$. By Chang–Fefferman duality, it is the case that $\|\mathbb{P}_{-,-}\varphi\|_{\mathrm{BMO}} \lesssim \|h_\varphi\|$.

In addition, due to the Hahn-Banach Theorem, and the inclusion $H^1 \subset L^1$, we can extend L to all of $L^1(\mathbb{T} \otimes \mathbb{T})$. Hence, there is a $\psi \in L^\infty$ with $\mathbb{P}_{-,-}\varphi = \mathbb{P}_{-,-}\psi$ and $\|\psi\|_\infty \lesssim \|h_\varphi\|$. $\quad\square$

We remark that in the one variable case, one achieves equality in (1.2). This is due to the fact that each $h \in H^1(\mathbb{D})$ can be factored as the product of functions in H^2, with equality of norms. In the present setting, one knows only weak factorization, that is the equality of $H^1(\mathbb{D} \otimes \mathbb{D})$ with the injective tensor product of $H^2(\mathbb{D} \otimes \mathbb{D})$ with itself.

To prove Corollary 1.4 we need the following lemma. Let S_j be the shift operator on $H^2(\mathbb{D} \otimes \mathbb{D})$ associated with multiplication by z_j, for $j = 1, 2$.

3.1 Lemma. *For all compact operators* $K \; : \; H^2_{+,+}(\mathbb{D} \otimes \mathbb{D}) \longrightarrow H^2_{-,-}(\mathbb{D} \otimes \mathbb{D})$, *we have* $\|KS_j^n S_{j'}^m\| \longrightarrow 0$, *for* $j, j' = 1, 2$.

Proof. It is enough to suppose that $j \neq j'$, for otherwise we simply have S_j^{n+m} and only have to deal with one of the multiplication operators, and the argument we give will also work. By symmetry we can suppose that $j = 1$ and $j' = 2$. It is also enough to deal with finite rank operators since we can approximate any compact operator by finite rank operators. We can actually check the claim for rank one operators, and only on a dense class of these operators. So take K to be defined by

$$K(f) = \langle f, g\rangle h \qquad \forall f \in H^2_{+,+}(\mathbb{D} \otimes \mathbb{D})$$

with $h \in H^2_{-,-}(\mathbb{D}\otimes\mathbb{D})$ and $g \in H^2_{+,+}(\mathbb{D}\otimes\mathbb{D})$ a polynomial of degree less than n in the z_1 variable and less than m in the z_2 variable. But $(S_1^*)^n(S_2^*)^m g = 0$, so we have that $KS_1^n S_2^m = 0$. $\quad\square$

Proof of Corollary 1.4. If $\mathbb{P}_{-,-}\varphi$ is in the BMO closure of $C(\mathbb{T}\otimes\mathbb{T})$, we can choose a polynomial ψ, antiholomorphic in each variable, such that $\|h_{\varphi-\psi}\|$ is small. But certainly h_ψ is finite rank, hence h_φ is the norm limit of finite rank operators. Hence it is compact.

Conversely, if h_φ is compact, then for any $\epsilon > 0$ we can choose n so large that $\|h_\varphi S_j^n\| < \epsilon$, for $j = 0, 1, 2$, where $S_0 := S_1 S_2$. Note that $h_\varphi S_j^n$ is also a Hankel operator associated to the function

$$\varphi_j := \overline{z}_j^n \mathbb{P}_{-,-} z_j^n \varphi, \quad j = 1, 2, \qquad \varphi_0 := \overline{z}_1^n \overline{z}_2^n \mathbb{P}_{-,-} z_1^n z_2^n \varphi.$$

5

Thus, by Theorem 1.1, φ_j has BMO$(\mathbb{D} \otimes \mathbb{D})$ norm at most a constant times ϵ. That is, the Hankel operator h_φ is well approximated in operator norm by the operator associated to the polynomial

$$\sum_{-n < m_1, m_2 < 0} \widehat{\varphi}(m_1, m_2) z_1^{m_1} z_2^{m_2} = \mathbb{P}_{-,-}\varphi + \varphi_0 - \varphi_1 - \varphi_2.$$

Thus, we see that $\mathbb{P}_{-,-}\varphi$ is in the BMO closure of $C(\mathbb{T} \otimes \mathbb{T})$. $\qquad\square$

Proof of Theorem 1.8. We prove the equivalence of the compactness of the commutator C_φ defined in (1.7) and $\varphi \in$ VMO, and prove the assertation that VMO$^* = H^1$ in the next section. Central to this proof is the characteriztion of the compactness of the Hankel operators that we have already given. While we discussed that proof on the circle, it has an equivalent formulation on the real line.[1]

Indeed, there are four relevant Hankel operators on $L^2(\mathbb{R} \otimes \mathbb{R})$. They are given as maps from $H^2_\varepsilon(\mathbb{R} \otimes \mathbb{R}) \longrightarrow H^2_{-\varepsilon}(\mathbb{R} \otimes \mathbb{R})$, where $\varepsilon \in \{\pm, \pm\}$, and $-\varepsilon$ is conjugate to ε. The definition is below, with M_φ being the operator of pointwise multiplication by φ.

$$H_{\varphi,\varepsilon}f = \mathbb{P}_{-\varepsilon}M_\varphi \; : \; H^2_\varepsilon(\mathbb{R} \otimes \mathbb{R}) \longrightarrow H^2_{-\varepsilon}(\mathbb{R} \otimes \mathbb{R}).$$

We have the fact that any of these operators is compact iff $\mathbb{P}_{-\varepsilon}\,\varphi \in$ VMO$(\mathbb{R} \otimes \mathbb{R})$. $L^2(\mathbb{R} \otimes \mathbb{R})$ is a sum of these Hardy spaces, and the commutator C_φ is a sum of these four Hankel operators. Thus, C_φ is compact iff each of the $H_{\varphi,\varepsilon}$ are compact iff $\varphi \in$ VMO$(\mathbb{R} \otimes \mathbb{R})$. $\qquad\square$

4 The Essential Norm of Little Hankel Operators

Proof of Theorem 1.5. We first show that (ii) and (iii) are equivalent. Suppose that $\varphi \in \mathcal{L}^\infty_{-,-} + C$, then we have $\varphi = \psi + g$ with $\psi \in \mathcal{L}^\infty_{-,-}$ and $g \in C$. Then for any $f \in H^2(\mathbb{D} \otimes \mathbb{D})$ we have

$$h_\varphi f = \mathbb{P}_{-,-}\varphi f = \mathbb{P}_{-,-}[gf + \psi f] = \mathbb{P}_{-,-}gf + \mathbb{P}_{-,-}\psi f = h_g f,$$

with the last line following because $\psi f \in \mathcal{L}^2_{-,-}(\mathbb{T} \otimes \mathbb{T})$ and $\mathbb{P}_{-,-}(\mathcal{L}^2_{-,-}(\mathbb{T} \otimes \mathbb{T})) = 0$. Thus giving that (ii) implies (iii).

Now assume that (iii) holds, we then have a function $g \in C(\mathbb{T} \otimes \mathbb{T})$ such that $h_\varphi = h_g$. Let $f \in H^2(\mathbb{D} \otimes \mathbb{D})$, then because $h_\varphi = h_g$ we have

$$\mathbb{P}_{-,-}((\varphi - g)f) = 0 \qquad \forall f \in H^2(\mathbb{D} \otimes \mathbb{D}).$$

Letting $f = 1$ we have that $\varphi - g \in \mathcal{L}^2_{-,-}(\mathbb{T} \otimes \mathbb{T})$, but we also have that $\varphi - g \in L^\infty(\mathbb{T} \otimes \mathbb{T})$, which implies that $\varphi - g \in \mathcal{L}^\infty_{-,-}(\mathbb{T} \otimes \mathbb{T})$, or $\varphi \in \mathcal{L}^\infty_{-,-} + C$.

Now we show that (i) and (ii) are equivalent. But this follows immediately from Theorem 1.6. This is because h_φ is compact if and only if $\|h_\varphi\|_e = 0$. But if $\|h_\varphi\|_e = 0$, then

---

[1]In fact, the paper of Ferguson and Lacey [5] is phrased on the real line, making certain simplifications for that proof available.

by Theorem 1.6 we have that $\mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-} + C) = 0$ and so $\varphi \in \mathcal{L}^\infty_{-,-} + C$. Conversely, if $\varphi \in \mathcal{L}^\infty_{-,-} + C$, then $\mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-} + C) = 0$. But by Theorem 1.6 we have $\|h_\varphi\|_e = 0$, or h_φ is compact. $\qquad\square$

Our proof of Theorem 1.6 is heavily influenced by the presentation of Hartman's Theorem in V. Peller's book [8]. We will need a few simple lemmas in the course of the proof of the theorem.

4.1 Lemma. *If $\psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T} \otimes \mathbb{T})$ and $\varphi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$ then $h_\varphi = h_{\varphi+\psi}$.*

Proof. Let $f \in H^2(\mathbb{D} \otimes \mathbb{D})$. Then

$$h_{\varphi+\psi}f = \mathbb{P}_{-,-}(\psi + \varphi)f = \mathbb{P}_{-,-}\psi f + \mathbb{P}_{-,-}\varphi f = \mathbb{P}_{-,-}\varphi f = h_\varphi f,$$

with the second to last inequality following since $\psi f \in \mathcal{L}^2_{-,-}(\mathbb{T} \otimes \mathbb{T})$ and $\mathbb{P}_{-,-}(\mathcal{L}^2_{-,-}(\mathbb{T} \otimes \mathbb{T})) = 0$. $\qquad\square$

4.2 Lemma. *Let $\varphi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$. Then*

$$\|h_\varphi\| \approx \inf\{\|\varphi - \psi\|_\infty : \psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T} \otimes \mathbb{T})\} := \mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-}).$$

This is the natural extenstion to the bi-disk of the fact in one complex variable that one can approximate the norm of a Hankel operator by the distance of its symbol from $H^\infty(\mathbb{D})$.

Proof. Clearly if $\varphi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$ and $\psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T} \otimes \mathbb{T})$ then

$$\|h_\varphi\| = \|h_{\varphi-\psi}\| \le \|\varphi - \psi\|_\infty,$$

and so

$$\|h_\varphi\| \le \mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-}).$$

On the other hand, Nehari's Theorem on the bi-disk, Theorem 1.1 implies that

$$\|\varphi - \psi\|_\infty \lesssim \|h_{\varphi-\psi}\| = \|h_\varphi\|$$

and so $\mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-}) \lesssim \|h_\varphi\|$. This proves the lemma. $\qquad\square$

We are also going to need a characterization of the space $\mathcal{L}^\infty_{-,-} + C$. Recall this is the space of functions $\varphi \in L^\infty(\mathbb{T} \otimes \mathbb{T})$ that have a decomposition of the form $\psi + g$ with $\psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T} \otimes \mathbb{T})$ and $g \in C(\mathbb{T} \otimes \mathbb{T})$. Similar to the one-variable case we have the following theorem.

4.3 Theorem. *$\mathcal{L}^\infty_{-,-} + C$ is a closed subspace of $L^\infty_{-,-}(\mathbb{T} \otimes \mathbb{T})$, and moreover*

$$\mathcal{L}^\infty_{-,-} + C = \mathrm{clos}_{L^\infty}\left(\bigcup_{n,m=0}^{\infty} \overline{z}_1^n \overline{z}_2^m \mathcal{L}^\infty_{-,-}(\mathbb{T} \otimes \mathbb{T})\right).$$

This result is slightly different than what one would find in one complex variable. In one variable, the analog of this space is $H^\infty + C$, which is in fact a sub-algebra of $L^\infty(\mathbb{T})$. In higher dimensions, $\mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})$ is not closed under multiplication as $H^\infty(\mathbb{D}\otimes\mathbb{D})$ is, so $\mathcal{L}^\infty_{-,-} + C$ will not be a sub-algebra. This is also a remnant of the fact we are working with little Hankel operators. To prove this theorem, we will need one more lemma.

4.4 Lemma. *Let* $C_{\mathcal{L}}(\mathbb{T}\otimes\mathbb{T}) := \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T}) \cap C(\mathbb{T}\otimes\mathbb{T})$ *and* $\varphi \in C(\mathbb{T}\otimes\mathbb{T})$. *Then*

$$\mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-}) = \mathrm{dist}_{L^\infty}(\varphi, C_{\mathcal{L}}).$$

Proof. Since $C_{\mathcal{L}}(\mathbb{T}\otimes\mathbb{T}) \subset \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})$ we trivially have that

$$\mathrm{dist}_{L^\infty}(\varphi, \mathcal{L}^\infty_{-,-}) \le \mathrm{dist}_{L^\infty}(\varphi, C_{\mathcal{L}}).$$

We are going to use the harmonic extension of functions in $L^\infty(\mathbb{T}\otimes\mathbb{T})$ to the bi-disk. The extension will also be denoted by the function element. Finally, let $h_r(\xi) = h(r\xi)$, $\xi \in \mathbb{T}\otimes\mathbb{T}$. Let $\psi \in \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})$. Since $\varphi \in C(\mathbb{T}\otimes\mathbb{T})$ we have

$$
\begin{aligned}
\|\varphi - \psi\|_\infty &\ge \lim_{r\to 1}\|(\varphi - \psi)_r\|_\infty \\
&\ge \lim_{r\to 1}(\|\varphi - \psi_r\|_\infty - \|\varphi - \varphi_r\|_\infty) \\
&= \lim_{r\to 1}\|\varphi - \psi_r\|_\infty.
\end{aligned}
$$

But this then shows that $\|\varphi - \psi\|_\infty \ge \mathrm{dist}_{L^\infty}(\varphi, C_{\mathcal{L}})$ for any $\psi \in \mathcal{L}^\infty_{-,-}$, proving the lemma. $\qquad\square$

We can now prove Theorem 4.3.

Proof. By Lemma 4.4 we have that $C/C_{\mathcal{L}}$ has an isometric embedding in $L^\infty/\mathcal{L}^\infty_{-,-}$ and can thus be considered as a closed subspace of $L^\infty/\mathcal{L}^\infty_{-,-}$. Let $\rho : L^\infty \to L^\infty/\mathcal{L}^\infty_{-,-}$ be the natural quotient map. Then we have that $\mathcal{L}^\infty_{-,-} + C = \rho^{-1}(C/C_{\mathcal{L}})$. This follows from the fact that we have a subspace and are looking at the quotient map. The proof is the same as in the one variable case. See [8] for details.

Finally, $\mathcal{L}^\infty_{-,-} + C = \mathrm{clos}_{L^\infty}\left(\cup_{n,m=0}^\infty \overline{z}_1^n \overline{z}_2^m \mathcal{L}^\infty_{-,-}(\mathbb{T}\otimes\mathbb{T})\right)$. This follows since the continuous functions on $\mathbb{T}\otimes\mathbb{T}$ can be uniformly approximated by polynomials in z_1, z_2 and their conjugates. $\qquad\square$

Proof of Theorem 1.6. Let $K : H^2_{+,+}(\mathbb{D}\otimes\mathbb{D}) \to H^2_{-,-}(\mathbb{D}\otimes\mathbb{D})$ be a compact operator. Then we want to estimate $\|h_\varphi - K\|$ from below. Let S_j be multiplication by the variable z_j. Using that S_j is a contraction and properties of norms we have

$$
\begin{aligned}
\|h_\varphi - K\| &\ge \|(h_\varphi - K)S_1^n S_2^m\| \\
&\ge \|h_\varphi S_1^n S_2^m\| - \|KS_1^n S_2^m\| \\
&= \|h_{z_1^n z_2^m \varphi}\| - \|KS_1^n S_2^m\|
\end{aligned}
$$

8

$$\gtrsim \quad \text{dist}_{L^\infty}(\varphi, \overline{z}_1^n \overline{z}_2^m \mathcal{L}_{-,-}^\infty) - \|KS_1^n S_2^m\|$$
$$\geq \quad \text{dist}_{L^\infty}(\varphi, \mathcal{L}_{-,-}^\infty + C) - \|KS_1^n S_2^m\|$$

We used the fact that $\|h_\varphi\| \approx \text{dist}_{L^\infty}(\varphi, \mathcal{L}_{-,-}^\infty)$ as shown in Lemma 4.2 and the characterization of $\mathcal{L}_{-,-}^\infty + C$ given in Theorem 4.3. Now by Lemma 3.1 we have as $n, m \to \infty$ that

$$\|KS_1^n S_2^m\| \to 0.$$

So $\|h_\varphi - K\| \gtrsim \text{dist}_{L^\infty}(\varphi, \mathcal{L}_{-,-}^\infty + C)$ for any compact operator K. This then gives

$$\|h_\varphi\|_e \gtrsim \text{dist}_{L^\infty}(\varphi, \mathcal{L}_{-,-}^\infty + C).$$

To prove the other inequality, begin by supposing that g is a trigonometric polynomial. Then h_g is a compact (finite rank) operator. So for any $g \in C(\mathbb{T} \otimes \mathbb{T})$ the operator h_g is compact. Then we have

$$\|h_\varphi\|_e \leq \inf_{g \in C} \|h_\varphi - h_g\| = \inf_{g \in C} \|h_{\varphi-g}\|.$$

By Lemma 4.2 we then have

$$\|h_\varphi\|_e \lesssim \inf_{g \in C, \psi \in \mathcal{L}_{-,-}^\infty} \|\varphi - g - \psi\| \lesssim \text{dist}_{L^\infty}(\varphi, \mathcal{L}_{-,-}^\infty + C).$$

Combining these two estimate we have that $\|h_\varphi\|_e \approx \text{dist}_{L^\infty}(\varphi, \mathcal{L}_{-,-}^\infty + C)$. $\qquad\square$

5 VMO and Carleson Measures

We state an equivalent form of the definition of $\text{VMO}(\mathbb{R} \otimes \mathbb{R})$ in terms of Carleson measures and, in particular, in a variant of (2.1).

5.1 Proposition. *Fix a choice of wavelet w. A function b is in $\text{VMO}(\mathbb{R} \otimes \mathbb{R})$ iff any of the conditions below hold.*

(i) b is in the closure, in BMO norm, of all finite linear combinations of

$$\{w_R : R \in \mathcal{D} \times \mathcal{D}\}.$$

(ii) $b \in \text{BMO}(\mathbb{R} \otimes \mathbb{R})$, and writing $R = R_1 \times R_2$ for a rectangle R,

$$\lim_{N \to \infty} \Big\| \sum_{\substack{R \in \mathcal{D} \times \mathcal{D} \\ |\log|R_1|| + |\log|R_2|| > N}} \langle b, w_R \rangle w_R \Big\|_{\text{BMO}(\mathbb{R} \otimes \mathbb{R})} = 0,$$

$$\lim_{N \to \infty} \Big\| \sum_{\substack{R \in \mathcal{D} \times \mathcal{D} \\ R \not\subset \{|x| < N\}}} \langle b, w_R \rangle w_R \Big\|_{\text{BMO}(\mathbb{R} \otimes \mathbb{R})} = 0.$$

These two conditions are independent of the choice of wavelet basis.

Set $FW(w)$ to be the linear space of finite linear combinations of $\{w_R : R \in \mathcal{D} \times \mathcal{D}\}$. Our first lemma is

5.2 Lemma. *For any two choices of w, w',*

$$\text{clos}_{\text{BMO}}\text{FW}(w) = \text{clos}_{\text{BMO}}\text{FW}(w').$$

Observe that the space $\text{clos}_{\text{BMO}}\text{FW}(w)$ is invariant under dilations by factors of 2. And, under our assumptions on the wavelets, we have

$$\sum_R |\langle w', w_R \rangle| < \infty.$$

This fact clearly implies that each wavelet $w'_R \in \text{clos}_{\text{BMO}}\text{FW}(w)$, and moreover that the same is true of each element of $FW(w')$. Thus, the lemma is immediate. This frees us to make particular choices for w in different parts of our proof. In addition, we suppress the explicit choice of wavelet in our notation.

It is a routine matter to verify that $b \in \text{clos}_{\text{BMO}}\text{FW}$ iff it satisfies condition (ii) of Proposition 5.1.

Let us see condition (i) of the proposition, that is

$$\text{clos}_{\text{BMO}}C_0^\infty = \text{clos}_{\text{BMO}}FW.$$

We are free to choose the wavelet to be smooth and have compact spatial support, in which case it is clear that $FW \subset \text{clos}_{\text{BMO}}C_0^\infty$. And so we need only argue for the reverse inclusion. But it is very easy to verify that a function in C_0^∞ satisfies condition (ii) of the proposition. In fact, this verification depends upon the estimates below, valid for all $f \in C_0^\infty$, with constants that depend upon the choice of f.

$$|\langle f, w_R \rangle| \lesssim \begin{cases} |R|^{3/2}, & |R_1| + |R_2| < 1 \\ \frac{|R_1|}{\sqrt{|R_2|}}, & |R_1| < 1 < |R_2| \\ |R|^{-1/2}, & |R_1|, |R_2| > 1. \end{cases}$$

And so, a function in C_0^∞ can be well approximated in BMO norm by finite sums of wavelets.

We address the equality $\text{VMO}^* = H^1$. H^1 and BMO duality shows that $H^1 \subset \text{VMO}^*$, and so we should show the reverse containment. But duality and $\text{VMO} = \text{clos}_{\text{BMO}}FW$ also shows that for $f \in FW$,

$$\sup_{\substack{b \in \text{VMO} \\ \|b\|_{\text{BMO}}=1}} |\langle f, b \rangle| \geq c\|f\|_{H^1}.$$

So to conclude the identity, it would be enough to know that $\text{clos}_{H^1}FW = H^1$. This equality follows from one of the several equivalent definitions of H^1 that have been established by Chang and Fefferman. In particular, we have

$$\|f\|_{H^1} \approx \left\| \left[\sum_{R \in \mathcal{R}} \frac{|\langle f, w_R \rangle|^2}{|R|} \mathbf{1}_R \right]^{1/2} \right\|_1.$$

And this equivalence proves that $\text{clos}_{H^1}FW = H^1$.

References

[1] Gérard Bourdaud, *Remarques sur certains sous-espaces de* BMO($\mathbb{R}^n$) *et de* bmo($\mathbb{R}^n$), Ann. Inst. Fourier (Grenoble) **52** (2002), 1187–1218. 1 927 078 (French)

[2] Sun-Yung A. Chang and Robert Fefferman, *Some recent developments in Fourier analysis and H^p-theory on product domains*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 1–43. MR 86g:42038

[3] ———, *A continuous version of duality of H^1 with BMO on the bidisc*, Ann. of Math. (2) **112** (1980), 179–201. MR 82a:32009

[4] R. R. Coifman, R. Rochberg, and Guido Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), 611–635. MR 54 #843

[5] Sarah H. Ferguson and Michael T. Lacey, *A characterization of product BMO by commutators*, Acta Math. **189** (2002), 143–160. 1 961 195

[6] Sarah H. Ferguson and Cora Sadosky, *Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures*, J. Anal. Math. **81** (2000), 239–267. MR 2001h:47040

[7] Zeev Nehari, *On bounded bilinear forms*, Ann. of Math. (2) **65** (1957), 153–162. MR 18,633f

[8] Vladimir V. Peller, *Hankel operators and their applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003, ISBN 0-387-95548-8. 1 949 210

[9] Donald Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. **207** (1975), 391–405. MR 51 #13690

Michael T. Lacey
School of Mathematics
Georgia Institute of Technology
Atlanta GA 30332
lacey@math.gatech.edu
http://www.math.gatech.edu/~lacey

Erin Terwilleger
Department of Mathematics
University of Connecticut
Storrs, CT 06269-3009
terwilleger@math.uconn.edu
http://www.math.uconn.edu/~terwilleger

Brett D. Wick
Department of Mathematics
Brown University
Providence, RI 02912
bwick@math.brown.edu
http://www.math.brown.edu/~bwick